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Effective Lagrangian for the pseudoscalars interacting with photons in the presence of a background electromagnetic field

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We attempt to evaluate the effective Lagrangian for a classical background field interacting with the vacuum of two quantum fields. The integration of one of the quantum fields in general leads to a non-local term in the effective Lagrangian and thus becomes intractable during the integration of the other quantum field. We show that $\phi F\tilde{K}$ interaction is an exception. We present the complete calculation for the evaluation of the effective Lagrangian for a pseudoscalar field interacting with photons in the presence of a background electromagnetic field. Expression for the probability of the vacuum breaking down into a pseudoscalar-photon pair is evaluated. We conclude that the gradient of an electric field beyond a certain threshold can give rise to pseudoscalar-photon pair production.

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I. INTRODUCTION

Consider an interaction involving three fields. Let us approximate one of the fields to be a classical background field. We address to the following question: What will be the effective Lagrangian for the classical background field interacting with the vacuum of TWO quantum fields? This task is achieved for the $\phi F\tilde{K}$ interaction by integrating out the F and ϕ fields one after the other. The K field is treated as a classical background electromagnetic field.

The evaluation of the effective Lagrangian for a classical background field interacting with the vacuum of a quantum field originated in the works of W. Heisenberg and H. Euler [1] and V. Weisskopf [2]. An explicit expression for the effective Lagrangian for the electromagnetic field interacting with the vacuum of the fermion field (by integrating out the fermion field in the QED Lagrangian) was evaluated by Julian Schwinger in [3]. Schwinger introduced the proper time method for this purpose which involved solving for the 'dynamics' of a 'particle' with its space-time coordinates evolving with respect to a proper time variable. Schwinger's result has been reproduced using other techniques by various authors [4–14]. Systematic analysis of the formalisms and references to related works can be found in [15,16]. Schwinger mechanism has attracted considerable attention because of its wide range of applicability. Recently in the SLAC E144 experiment electron positron pairs by Schwinger mechanism are reported to have been produced in the collision of intense laser beams with a high energy electron beam [17]. Schwinger mechanism has been used to study particle creation by an external gravitational field to understand their implications on Cosmological models [18]. There is a strong conviction that the Hawking's theory of particle creation by black holes [19] can be understood using the Schwinger mechanism. In [22] Schwinger mechanism has been used to put bounds on the masses of particles.

Schwinger's result was derived from the QED Lagrangian for the case of a constant classical electromagnetic field. Various generalizations and extensions to the Schwinger's result have been attempted by different authors. Generalization of the Schwinger's result

for various other interactions involving two fields was done in [7]. Generalization for non abelian fields in the context of the QCD vacuum breaking down into quark antiquark pairs in the presence of background colour fields was studied in [20]. Schwinger mechanism has been extended to include finite temperature effects in [21]. Extension for the case where the background electromagnetic field is confined to a finite volume has been studied in [12].

In this work we evaluate the effective Lagrangian for a classical background electromagnetic field K interacting with the vacuum of the quantized pseudoscalar field ϕ and the quantized electromagnetic field F. The interaction term reads as $\phi F \tilde{K}$. The evaluation of the effective Lagrangian involves the integration of the ϕ and F fields one after the other. The effective Lagrangian for the $\phi F \tilde{K}$ interaction has been earlier evaluated in [22], where the emphasis has been to use the result to put a bound on the mass of the pseudoscalar particle called axion. The calculation in [22] for the evaluation of the effective Lagrangian is not completely satisfactory because the integration of the F field in the calculation is qualitatively evaluated using Feynman diagram techniques. The main goal of this paper is to complete this gap by giving the complete calculation for the integration of the F field. This part of the calculation is very important if one intends to generalize the result for other interactions.

In section II we formulate the problem and introduce the formalism used. In section III we evaluate the F integral. We shall explicitly show that the contribution to the effective Lagrangian obtained after the integration of the F field is not non-local because one of the integrals leads to a delta function. In section IV we evaluate the integration involving the ϕ field to arrive at the expression for the effective Lagrangian under consideration. Here we use the Schwinger's proper time method to evaluate the integration of the ϕ field. We show that the result obtained by using the proper time method exactly matches with the result obtained in [22] using the Green's function method which was more in the spirit of Brown and Duff's paper [7]. The complete calculation of the effective Lagrangian can be summarized as:

$$\mathcal{L}[\phi, F, K] = -\frac{1}{4}K^2 - \frac{1}{4}F^2 + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{1}{2}g\phi F\tilde{K}$$

$$\Big\downarrow \quad F \text{ integration}$$

$$\mathcal{L}'_{eff}[\phi, K] = -\frac{1}{4}K^2 + \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{8}g^2K^2\phi^2$$

$$\Big\downarrow \quad \phi \text{ integration}$$

$$\mathcal{L}_{eff}[K] = -\frac{1}{4}K^2 + \frac{1}{32\pi^2}\int_0^\infty \frac{ds}{s^3}e^{-m^2 s}\left[\left\{\det\left(\frac{gK_1 s}{\sin gK_1 s}\right)\right\}^{\frac{1}{2}} e^{-x^2 gK_1 \tan\left(\frac{gK_1 s}{2}\right)} - 1\right]$$

where K_1 is the gradient of the electromagnetic field. As a corollary, in section V we find the expression for the probability of the vacuum to breakdown into pseudoscalar photon pairs in the presence of a classical background electromagnetic field. This is related to the imaginary part of the effective Lagrangian. We conclude that the gradient of the electric field beyond a certain threshold can give rise to pair production. This is unlike the QED case where a constant electric field contributes to pair production.

II. FORMULATION OF THE PROBLEM

The Lagrangian for a pseudoscalar field $\phi(x)$ interacting with a massless vector field $A'_\mu(x)$ is given by

$$\mathcal{L}[\phi, A'] = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + \frac{1}{4}g\phi F'_{\mu\nu}\tilde{F'}^{\mu\nu}. \tag{1}$$

The factor of $\frac{1}{2}$ in the kinetic term of the pseudoscalar field is to take care of the fact that we are not treating ϕ as a complex field. In eqn. (1) $F'_{\mu\nu}(x)$ is the field tensor corresponding

to ${A'}_{\mu}(x)$ and $\tilde{F'}^{\mu\nu}(x)$ is the dual tensor corresponding to ${F'}_{\mu\nu}(x)$ given by

$$\tilde{F'}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}{F'}_{\alpha\beta}. \tag{2}$$

We shall now show that under the circumstance when the A' field can be treated as a superposition of two fields the $\phi F'\tilde{F}'$ interaction can be approximated to $\phi F\tilde{K}$ interaction. Let us consider the massless vector field $A'(x)$ to be a superposition of two independent massless vector fields given by

$$A'^{\mu}(x) \equiv A^{\mu}(x) + a^{\mu}(x), \tag{3}$$

which amounts to

$$F'^{\mu\nu}(x) = F^{\mu\nu}(x) + K^{\mu\nu}(x), \tag{4}$$

where $F^{\mu\nu}(x)$ is the field tensor corresponding to the field $A^{\mu}(x)$, and $K^{\mu\nu}(x)$ is the field tensor corresponding to the field $a^{\mu}(x)$. Let the energy corresponding to the $a^{\mu}(x)$ field be E_a. Because of the interaction it has with the $A^{\mu}(x)$ and $\phi(x)$ field the energy of the $a^{\mu}(x)$ field gets modified to $E_a + E_{back}$. Let us *assume* that the correction E_{back} is negligible compared to E_a. That is we make the assumption

$$E_{back} \ << \ E_a. \tag{5}$$

In the above approximation we are neglecting the backreaction of the $A^{\mu}(x)$ and $\phi(x)$ field on the $a^{\mu}(x)$ field. This is analogous to the case in quantum electrodynamics when we neglect the radiation effects (back reaction of the electron on the electromagnetic field). Next we notice that neglecting backreaction effects (radiation effects) amounts to treating one of the fields as a classical field because radiation effects becomes significant only in the quantum regime (for distances below $\frac{\hbar}{mc}$, where m is the mass of the particle under consideration). That is to say, if we are neglecting the backreaction effects we can as well treat the field to be a classical field. Thus we shall interpret $a^{\mu}(x)$ as a classical background electromagnetic field. $A^{\mu}(x)$ and $\phi(x)$ will be treated as quantum fields.

Including the above considerations in the interaction term of the Lagrangian in eqn. (1) we have

$$\mathcal{L}_{int}[\phi, A, a] = \frac{1}{4} g\phi (F_{\mu\nu} + K_{\mu\nu})(\tilde{F}^{\mu\nu} + \tilde{K}^{\mu\nu})$$
$$= \frac{1}{4} g\phi F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2} g\phi F_{\mu\nu}\tilde{K}^{\mu\nu} + \frac{1}{4} g\phi K_{\mu\nu}\tilde{K}^{\mu\nu}, \tag{6}$$

where we have used the relation $K_{\mu\nu}\tilde{F}^{\mu\nu} = F_{\mu\nu}\tilde{K}^{\mu\nu}$. Since we shall be interested in the interaction of all the three fields together, i.e. the pseudoscalar field and the quantum electromagnetic field in the presence of a classical background electromagnetic field, we shall *ignore* the $\phi F\tilde{F}$ and the $\phi K\tilde{K}$ terms. This is done to simplify the calculations. Thus we have the Lagrangian to be

$$\mathcal{L}[\phi, A, a] = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2} m^2\phi^2 + \frac{1}{2} g\phi F_{\mu\nu}\tilde{K}^{\mu\nu}, \tag{7}$$

where $\tilde{K}$ involves the classical background electromagnetic field $a_\mu(x)$. The corresponding action is given by

$$S[\phi, A, a] = \int d^4x \ \mathcal{L}[\phi, A, a]. \tag{8}$$

The vacuum to vacuum transition amplitude is given by

$$W = \frac{\int \mathcal{D}\phi \int \mathcal{D}A \ e^{iS[\phi, A, a]}}{\int \mathcal{D}\phi \int \mathcal{D}A \ e^{iS_0[\phi, A, a]}} \tag{9}$$

where $S_0[\phi, A, a]$ is the Action in the absence of any interaction. We define the effective action corresponding to the approximation in eqn. (5) as

$$W = e^{iS_{eff}[a]}. \tag{10}$$

Note that the effective action is a function of the $a^\mu(x)$ field alone. Thus evaluation of the effective action will involve the integration of the ϕ field and the A field in eqn. (9). In section III we evaluate the integration of the A field and in section IV we evaluate the integral involving the ϕ field.

The probability for the vacuum to breakdown into pairs is given by

$$P = 1 - |W|^2$$
$$= 1 - \exp\left[-2\ \mathrm{Im}\ S_{eff}[a]\right]. \tag{11}$$

In section V we shall use this expression to calculate the probability for the vacuum to breakdown into pseudoscalar photon pairs in the presence of a classical background electromagnetic field.

III. EVALUATION OF THE A INTEGRATION

In this section we shall begin with eqn. (9) and evaluate the integral involving the A field. The A integral is first converted into a Gaussian integral. The contribution to the effective Lagrangian obtained after the evaluation of the Gaussian integral on the first look seems to be non-local. We shall explicitly show that this contribution is not non-local by showing that one of the integrals leads to a delta function. In [22] this part of the calculation was evaluated qualitatively using Feynman diagram techniques.

To begin with we rewrite eqn. (9) as

$$W = \frac{\int \mathcal{D}\phi\ e^{i\int d^4x\ \left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2\right]} \int \mathcal{D}A\ e^{i\int d^4x\ \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}g\phi F_{\mu\nu}\tilde{K}^{\mu\nu}\right]}}{\int \mathcal{D}\phi\ e^{i\int d^4x\ \left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2\right]} \int \mathcal{D}A\ e^{i\int d^4x\ \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]}}. \tag{12}$$

After integration by parts (x integral) the A integral in the numerator can be rewritten as

$$\int \mathcal{D}A\ e^{i\int d^4x\ \left[-\frac{1}{2}A_\mu(\partial^\mu\partial^\nu - g^{\mu\nu}\partial_\alpha\partial^\alpha)A_\nu + \left\{g\partial_\nu\left(\phi(x)\tilde{K}^{\mu\nu}(x)\right)\right\}A_\mu\right]}. \tag{13}$$

The Gaussian integral formula for a massless vector field is given by

$$\int \mathcal{D}A\ e^{i\int d^4x\ \left[-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) + J_\mu(x)A^\mu(x)\right]}$$
$$= e^{-\frac{1}{2}\int d^4x\ \int d^4x'\ J_\mu(x)G^{\mu\nu}(x-x')J_\nu(x')} \int \mathcal{D}A\ e^{i\int d^4x\ \left[-\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x)\right]} \tag{14}$$

where

$$G^{\mu\nu}(x-x') = \frac{i}{(2\pi)^4}\int d^4k\ \left[k^\mu k^\nu - g^{\mu\nu}k^2\right]^{-1} e^{-ik(x-x')} \tag{15}$$

Using eqn. (14) and eqn. (13) in eqn. (12) we have

$$W = \frac{\int \mathcal{D}\phi \; e^{i\int d^4x\left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{1}{2}m^2\phi^2\right]-\frac{1}{2}\int d^4x\int d^4x' \;\left[g\partial_\alpha\left(\phi(x)\tilde{K}^{\mu\alpha}(x)\right)\right]G_{\mu\nu}(x-x')\left[g\partial'_\beta\left(\phi(x')\tilde{K}^{\nu\beta}(x')\right)\right]}}{\int \mathcal{D}\phi \; e^{i\int d^4x\;\left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{1}{2}m^2\phi^2\right]}}. \quad (16)$$

Again by integration by parts we can show that the integral in the interaction term in the numerator is

$$\begin{aligned}\int d^4x\;\left[g\partial_\alpha\left(\phi(x)\tilde{K}^{\mu\alpha}(x)\right)\right]\int d^4x'\;G_{\mu\nu}(x-x')\left[g\partial'_\beta\left(\phi(x')\tilde{K}^{\nu\beta}(x')\right)\right]\\ = g^2\int d^4x\;\phi(x)\tilde{K}^{\mu\alpha}(x)\int d^4x'\;\left[\partial_\alpha\partial'_\beta G_{\mu\nu}(x-x')\right]\tilde{K}^{\nu\beta}(x')\phi(x'). \quad (17)\end{aligned}$$

Using eqn. (17) in eqn. (16) we have

$$W = \frac{\int \mathcal{D}\phi \; e^{i\int d^4x\left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{1}{2}m^2\phi^2-\frac{1}{2}g^2\int d^4x\int d^4x'\;\phi(x)M(x,x')\phi(x')\right]}}{\int \mathcal{D}\phi \; e^{i\int d^4x\;\left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{1}{2}m^2\phi^2\right]}} \quad (18)$$

where

$$M(x,x') \equiv \tilde{K}^{\mu\alpha}(x)\left[\partial_\alpha\partial'_\beta G_{\mu\nu}(x-x')\right]\tilde{K}^{\nu\beta}(x'). \quad (19)$$

Thus the effective Lagrangian obtained after integrating out the A field seems to have a non local contribution from $M(x,x')$. But we shall show that the $M(x,x')$ involves a delta function and thus show that the term is not non local.

We shall now show that $M(x,x')$ takes the following form in terms of the delta function.

$$M(x,x') = -\frac{i}{4}\tilde{K}^{\mu\nu}(x)\tilde{K}_{\mu\nu}(x')\delta^4(x-x'). \quad (20)$$

To arrive at the above expression we begin with eqn. (19) and using eqn. (15) we have

$$M(x,x') = \tilde{K}^{\mu\alpha}(x)\tilde{K}^{\nu\beta}(x')\frac{i}{(2\pi)^4}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x'^\beta}\int_{-\infty}^{+\infty} d^4k\;\left[k_\mu k_\nu - g_{\mu\nu}k^2\right]^{-1} e^{-ik(x-x')}. \quad (21)$$

Completing the differentiations with respect to x and x' and using

$$\left[k_\mu k_\nu - g_{\mu\nu}k^2\right]^{-1} = \frac{1}{k^4}(k_\mu k_\nu - g_{\mu\nu}k^2) \quad (22)$$

we have

$$M(x,x') = i\;\tilde{K}^{\mu\alpha}(x)\tilde{K}^{\nu\beta}(x')\frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty} d^4k\;(k_\mu k_\nu - g_{\mu\nu}k^2)\;\frac{k_\alpha k_\beta}{k^4}\;e^{-ik(x-x')}. \quad (23)$$

Only the even terms in the d^4k integral in the above expression give a non zero contribution. Thus we can evaluate the integral by replacing $k_\alpha k_\beta$ with $\frac{1}{2}g_{\alpha\beta}k^2$. Using these substitutions in eqn. (23) we have

$$M(x,x') = -\frac{i}{4}\tilde{K}^{\mu\nu}(x)\tilde{K}_{\mu\nu}(x')\frac{1}{(2\pi)^4}\int d^4k\ e^{-ik(x-x')}. \tag{24}$$

Using

$$\delta^4(x-x') = \frac{1}{(2\pi)^4}\int d^4k\ e^{-ik(x-x')} \tag{25}$$

in eqn. (24) we have the result

$$M(x,x') = -\frac{i}{4}\tilde{K}^{\mu\nu}(x)\tilde{K}_{\mu\nu}(x')\delta^4(x-x'). \tag{26}$$

Completing the x' integral after substituting eqn. (26) in eqn. (18) and then using the relation $\tilde{K}^{\mu\nu}(x)\tilde{K}_{\mu\nu}(x) = -K^{\mu\nu}(x)K_{\mu\nu}(x)$ we get

$$W = \frac{\int \mathcal{D}\phi\ e^{i\int d^4x\left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{1}{2}m^2\phi^2-\frac{1}{8}g^2\phi(x)K^{\mu\nu}(x)K_{\mu\nu}(x)\phi(x)\right]}}{\int \mathcal{D}\phi\ e^{i\int d^4x\ \left[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{1}{2}m^2\phi^2\right]}}. \tag{27}$$

It should be observed that the expression in the exponential of eqn. (27) is the effective Lagrangian for the case when both the ϕ and K fields are treated as classical background fields.

IV. EVALUATION OF THE ϕ INTEGRAL

In this section we shall begin with eqn. (27) and evaluate the integral involving the ϕ field. In [22] this was achieved by using the Green's function method adopted by Brown and Duff in [7]. Here we shall do the same using the proper time method introduced by Schwinger in [3]. Proper time method reduces the problem into an 'associated dynamical problem' with respect to a proper time variable. The effective Lagrangian in this method is given in terms of the trace of an evolution operator (evolving in the proper time). The evolution operator is determined by solving the 'heat equation' corresponding to the 'associated dynamical

problem'. We shall show that the result obtained by using the proper time method here exactly matches with the result obtained in [22] using the Green's function method.

To begin with we rewrite eqn. (27) as

$$e^{i\int d^4x\ \mathcal{L}_{eff}(x)} = \frac{\int \mathcal{D}\phi\ e^{i\int d^4x\ \phi(x)\left[-\frac{1}{2}\partial_\mu\partial^\mu - \frac{1}{2}m^2 - \frac{1}{8}g^2K^2(x)\right]\phi(x)}}{\int \mathcal{D}\phi\ e^{i\int d^4x\ \phi(x)\left[-\frac{1}{2}\partial_\mu\partial^\mu - \frac{1}{2}m^2\right]\phi(x)}} \tag{28}$$

where $K^2 = K^{\mu\nu}K_{\mu\nu}$. Using the Gaussian integral formula for a scalar field given by

$$\int \mathcal{D}\phi\ e^{i\int d^4x\ \phi(x)\ \mathrm{M}\ \phi(x)} = \text{const}\left[\det\ \mathrm{M}\right]^{-\frac{1}{2}} \tag{29}$$

where M is any operator and exploiting the identity $\det \mathrm{M} = \exp[\mathrm{Tr}\ \ln \mathrm{M}]$ we have

$$\int d^4x\ \mathcal{L}_{eff}(x) = \frac{i}{2}\ \mathrm{Tr}\ln\left\{\frac{1}{2}\partial_\mu\partial^\mu + \frac{1}{2}m^2 + \frac{1}{8}g^2K^2(x)\right\} - \frac{i}{2}\ \mathrm{Tr}\ln\left\{\frac{1}{2}\partial_\mu\partial^\mu + \frac{1}{2}m^2\right\}. \tag{30}$$

Thus the evaluation of the effective Lagrangian involves the determination of the trace of the logarithm of an operator. We shall now very briefly give the prescription for the determination of the trace of the logarithm of an operator. Let us begin with the integrals

$$\ln y = \int_1^y dt\ \frac{1}{t} \qquad \text{and} \qquad \frac{1}{t} = \int_0^\infty ds\ e^{-ts} \qquad t > 0. \tag{31}$$

Generalizing these integrals for an operator we have the integral representation for $\ln \mathrm{M}$ to be

$$\ln \mathrm{M} = -\int_0^\infty \frac{ds}{s}\ e^{-\mathrm{M}s}. \tag{32}$$

Taking the trace of the above integral on both sides and evaluating the trace in the position basis we have

$$\mathrm{Tr}\ln\mathrm{M} = -\int dx\ \int_0^\infty \frac{ds}{s}\ \left\{K(x,x';s)\right\}_{x=x'} \tag{33}$$

where $K(x,x';s) = \exp\left[-\mathrm{M}s\right]\delta(x-x')$. By differentiating $K(x,x';s) = \exp\left[-\mathrm{M}s\right]\delta(x-x')$ with respect to s on both sides we get the differential equation for $K(x,x';s)$ to be

$$\mathrm{M}\ K(x,x';s) = -\frac{\partial}{\partial s}K(x,x';s) \tag{34}$$

with the initial condition $K(x,x';0)=\delta(x-x')$. Thus $\mathrm{Tr}\ln\mathrm{M}$ is given in terms of the trace of an evolution operator evolving in the proper time s. The evolution operator is to be determined by solving the differential equation in eqn. (34).

Using the prescription in eqn. (33) in the expression for the effective Lagrangian in eqn. (30) we get

$$\mathcal{L}_{eff}(x)=\frac{i}{2}\int_0^\infty\frac{ds}{s}\left\{U(x,x';s)\right\}_{x=x'}-\frac{i}{2}\int_0^\infty\frac{ds}{s}\left\{U_0(x,x';s)\right\}_{x=x'} \tag{35}$$

where $U(x,x';s)$ and $U_0(x,x';s)$ satisfy the differential equations

$$\left\{\frac{1}{2}\partial_\mu\partial^\mu+\frac{1}{2}m^2+\frac{1}{8}g^2K^2(x)\right\}U(x,x';s)=-\frac{\partial}{\partial s}U(x,x';s) \tag{36}$$

and

$$\left\{\frac{1}{2}\partial_\mu\partial^\mu+\frac{1}{2}m^2\right\}U_0(x,x';s)=-\frac{\partial}{\partial s}U_0(x,x';s) \tag{37}$$

with the initial conditions $U(x,x';0)=\delta^4(x-x')$ and $U_0(x,x';0)=\delta^4(x-x')$. Thus the problem of evaluation of the effective Lagrangian reduces to solving the differential equations in eqn. (36) and eqn. (37) with the appropriate initial conditions. Eqn. (36) cannot be solved for a general $K^2(x)$. We thus expand $K_{\mu\nu}(x)$ using Taylor expansion and take only the leading order terms. Thus we can write

$$K_{\mu\nu}(x)=\mathrm{K_0}+\mathrm{K_1}\cdot(\mathrm{x}-\mathrm{x_0})+... \tag{38}$$

where $\mathrm{K_0}\equiv K_{\mu\nu}(x_0)$ and $\mathrm{K_1}\equiv\partial_\alpha K_{\mu\nu}(x_0)$. K_0 corresponds to the constant part of the electromagnetic field and K_1 is the gradient of the electromagnetic field. Let us choose x_0 to be the stationary point of $K^2(x)$ given by

$$x_0=\frac{K_0}{K_1}. \tag{39}$$

This choice corresponds to the saddle point approximation. Under this approximation the differential equation in eqn. (36) becomes

$$\left\{\frac{1}{2}\partial_\mu\partial^\mu+\frac{1}{2}m^2+\frac{1}{8}g^2K_1^{\mu\nu 2}x_\mu x_\nu\right\}U(x,x';s)=-\frac{\partial}{\partial s}U(x,x';s). \tag{40}$$

In appendix A we solve the above differential equation and get the solution as

$$U(x,x';s) = -i\frac{1}{4\pi^2}\frac{1}{s^2}e^{-\frac{m^2}{2}s}\left[\det\left(\frac{\frac{gK_1 s}{2}}{\sin\frac{gK_1 s}{2}}\right)\right]^{\frac{1}{2}} e^{l(x,x';s)} \tag{41}$$

where

$$l(x,x';s) = \frac{1}{4}(x^2 + x'^2)gK_1 \cot\left(\frac{gK_1 s}{2}\right) - \frac{1}{2}xx'gK_1 \mathrm{cosec}\left(\frac{gK_1 s}{2}\right). \tag{42}$$

The differential equation for $U_0(x,x';s)$ can also be very easily solved. The solution is

$$U_0(x,x';s) = -i\frac{1}{4\pi^2}\frac{1}{s^2}e^{-\frac{m^2}{2}s}e^{\frac{(x-x')^2}{2s}}. \tag{43}$$

From the solutions for $U(x,x';s)$ and $U_0(x,x';s)$ it is observed that

$$\lim_{K_0 \to\ 0}\ \lim_{K_1 \to\ 0} U(x,x';s) = U_0(x,x';s) \tag{44}$$

which implies that in the absence of any interaction the evolution operator reduces to that of a free field. Using the solutions for $U(x,x';s)$ and $U_0(x,x';s)$ (eqn. (41) and eqn. (43)) in the expression for the effective Lagrangian in eqn. (35) we get

$$\mathcal{L}_{eff}(x) = \frac{1}{32\pi^2}\int_0^\infty \frac{ds}{s^3}e^{-m^2 s}\left[\left\{\det\left(\frac{gK_1 s}{\sin gK_1 s}\right)\right\}^{\frac{1}{2}} \exp\left\{-x^2 gK_1 \tan\left(\frac{gK_1 s}{2}\right)\right\} - 1\right] \tag{45}$$

The above expression for the effective Lagrangian is best applicable for space time points around the stationary point $x = \frac{K_0}{K_1}$ because of the approximation made in eqn. (39).

V. PSEUDOSCALAR-PHOTON PAIR CREATION

The effective Lagrangian for the $\phi F\tilde{K}$ interaction in eqn. (45) when evaluated around the stationary point $x = \frac{K_0}{K_1}$ gives

$$\mathcal{L}^{\phi F\tilde{K}}_{eff}(K_0, K_1) = \frac{1}{32\pi^2}\int_0^\infty \frac{ds}{s^3}e^{-m^2 s}\left[\left\{\det\left(\frac{gK_1 s}{\sin gK_1 s}\right)\right\}^{\frac{1}{2}} e^{-l(K_0,K_1;s)} - 1\right] \tag{46}$$

where

$$l(K_0,K_1;s) = \frac{1}{2}g^2 {K_0}^2 s\ \frac{\tan\left(\frac{gK_1 s}{2}\right)}{\frac{gK_1 s}{2}} \tag{47}$$

The probability of pseudoscalar photon pair production is related to the effective Lagrangian for the $\phi F\tilde{K}$ interaction by the formula (using eqn. (11))

$$P^{\phi F\tilde{K}} = 1 - \exp\left[-2VT \ \mathrm{Im} \ \mathcal{L}_{eff}^{\phi F\tilde{K}}\right] \tag{48}$$

where V is the total volume of the space and T is the total time for which the system is observed. Note that VT should be centered about the stationary point $x = \frac{K_0}{K_1}$. The expression for the effective Lagrangian for the $\phi F\tilde{K}$ interaction in eqn. (46) involves the determinant of K_1. Determinant of an operator is equal to the product of its eigenvalues. Schwinger in [3] evaluated the eigenvalues of the electromagnetic field tensor $K_{\mu\nu}$ using a very simple and elegant trick. Using the basic relations satisfied by $K_{\mu\nu}$ he constructed an eigenvalue equation satisfied by the eigenvalues of $K_{\mu\nu}$ to be

$$\lambda^4 + \left(\mathbf{B}^2 - \mathbf{E}^2\right)\lambda^2 - \left(\mathbf{B}\cdot\mathbf{E}\right)^2 = 0 \tag{49}$$

The roots of the above equation which are the eigenvalues of $K_{\mu\nu}$ are $+\lambda_r$, $-\lambda_r$, $+i\lambda_i$, $-i\lambda_i$, where

$$\lambda_r = \sqrt{R}\ \cos\frac{\theta}{2} \qquad \text{and} \qquad \lambda_i = \sqrt{R}\ \sin\frac{\theta}{2}. \tag{50}$$

R and θ are constructed as

$$Re^{i\theta} \equiv \left(\mathbf{E}^2 - \mathbf{B}^2\right) + i2\left(\mathbf{E}\cdot\mathbf{B}\right). \tag{51}$$

Thus we have

$$R^2 = \left(\mathbf{E}^2 - \mathbf{B}^2\right)^2 + 4\left(\mathbf{E}\cdot\mathbf{B}\right)^2 \qquad \text{and} \qquad \tan\theta = \frac{2\left(\mathbf{E}\cdot\mathbf{B}\right)}{\mathbf{E}^2 - \mathbf{B}^2}. \tag{52}$$

Thus one pair of eigenvalues is always real and the other pair is always imaginary. This observation is very crucial because we shall soon see that only the real pair of eigenvalues contribute to the imaginary part of the effective Lagrangian. Note that if the electromagnetic field consists of a pure electric field then $\lambda_r = |\mathbf{E}|$ and $i\lambda_i = 0$, and if it consists of a pure magnetic field then $\lambda_r = 0$ and $i\lambda_i = i\,|\mathbf{B}|$.

Using the above results in eqn. (46) we have

$$\text{Im } \mathcal{L}_{eff}^{\phi F\tilde{K}}(\lambda_r, \lambda_i) = \frac{1}{32\pi^2}\text{Im } \int_0^\infty \frac{ds}{s^3} e^{-m^2 s} \left[\frac{g\lambda_r^1 s}{\sin \ g\lambda_r^1 s} \ \frac{g\lambda_i^1 s}{\sinh \ g\lambda_i^1 s} \ e^{-l(\lambda_r, \lambda_i; s)} - 1 \right] \tag{53}$$

where

$$l(\lambda_r, \lambda_i; s) = g^2 \lambda_r^{0^2} s \ \frac{\tan\left(\frac{g\lambda_r^1 s}{2}\right)}{\frac{g\lambda_r^1 s}{2}} \ + \ g^2 \lambda_i^{0^2} s \ \frac{\tan\left(\frac{g\lambda_i^1 s}{2}\right)}{\frac{g\lambda_i^1 s}{2}} \tag{54}$$

where λ^0 and λ^1 are the coefficients of the Taylor expansion series of λ. That is

$$\begin{aligned} \lambda_r &= \lambda_r^0 + \lambda_r^1 \cdot (x - x_0) \\ \lambda_i &= \lambda_i^0 + \lambda_i^1 \cdot (x - x_0) \end{aligned} \tag{55}$$

$\text{Im } \mathcal{L}_{eff}^{\phi F\tilde{K}}$ involves integrating the imaginary part of the above integral. This integral is not analytic in λ_r. Thus we evaluate $\text{Im } \mathcal{L}_{eff}^{\phi F\tilde{K}}$ by the prescription

$$\text{Im } \mathcal{L}_{eff}^{\phi F\tilde{K}}(\lambda_r, \lambda_i) = \lim_{\epsilon \longrightarrow 0} \ \text{Im } \mathcal{L}_{eff}^{\phi F\tilde{K}}(\lambda_r - i\epsilon, \lambda_i). \tag{56}$$

The evaluation of the s integral is achieved by extending s to the complex plane. The integrand has poles at points satisfying $g\lambda_r^1 s = n\pi$ and $g\lambda_i^1 s = in\pi$ for integer n. The prescription in eqn. (56) is equivalent to shifting the poles corresponding to λ_r^1 below the real line. The contour is chosen to pass along the real axis. Using these we get

$$\text{Im } \mathcal{L}_{eff}^{\phi F\tilde{K}}(\lambda_r, \lambda_i) = \frac{1}{32\pi^4} g^2 \lambda_r^{1^2} \sum_{n=0}^\infty \frac{1}{n^2} \ \exp\left(-\frac{n\pi m^2}{g\lambda_r^1}\right) \ \frac{n\pi \frac{\lambda_i^1}{\lambda_r^1}}{\sinh n\pi \frac{\lambda_i^1}{\lambda_r^1}} \ e^{-il(\lambda_r, \lambda_i)} \tag{57}$$

where

$$l(\lambda_r, \lambda_i) = n\pi g \ \frac{\lambda_r^{0^2}}{\lambda_r^1} \ \frac{\tanh\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}} \ + \ n\pi g \ \frac{\lambda_i^{0^2}}{\lambda_r^1} \ \frac{\tanh\left(\frac{n\pi}{2}\frac{\lambda_i^1}{\lambda_r^1}\right)}{\frac{n\pi}{2}\frac{\lambda_i^1}{\lambda_r^1}}. \tag{58}$$

The above expression for the imaginary part of the effective Lagrangian for the $\phi F\tilde{K}$ interaction in conjunction with eqn. (11) gives the expression for the probability for the vacuum to breakdown into pseudoscalar photon pairs in the presence of a classical background electromagnetic field. The corresponding expression for the $\phi^2 A^2$ interaction calculated by Schwinger in [3] in the above notation will be

$$\mathrm{Im}\ \mathcal{L}_{eff}^{\phi^2 A^2}(\lambda_r, \lambda_i) = \frac{1}{32\pi^4} e^2 \lambda_r^{02} \sum_{n=0}^{\infty} \frac{1}{n^2}\ \exp\left(-\frac{n\pi m_e^2}{e\lambda_r^0}\right)\ \frac{n\pi\frac{\lambda_i^0}{\lambda_r^0}}{\sinh n\pi\frac{\lambda_i^0}{\lambda_r^0}} \tag{59}$$

where m_e is the mass of the electron and e is the charge of the electron. The difference in the factor of 2 with the Schwinger's result in [3] is because we do not consider our scalar field to be a complex field.

Observe that for the $\phi^2 A^2$ interaction λ_r^0 (constant electric field) is responsible for pair creation. λ_r^0 (constant electric field) does not contribute to pair creation in the case of $\phi F\tilde{K}$ interaction. For the $\phi F\tilde{K}$ interaction λ_r^1 (gradient of electric field) is necessary for pair creation. Thus we conclude that for the $\phi F\tilde{K}$ interaction we need a varying electric field (in space or time) for pair creation.

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APPENDIX A: SOLUTION TO THE DIFFERENTIAL EQUATION IN EQN. (40)

Here we solve the differential equation

$$\left\{\frac{1}{2}\partial_\mu\partial^\mu + \frac{1}{2}m^2 + \frac{1}{2}\lambda_{\mu\nu}^2 x^\mu x^\nu\right\} U(x, x'; s) = -\frac{\partial}{\partial s} U(x, x'; s) \tag{A1}$$

with the initial condition $U(x, x'; 0) = \delta^4(x - x')$. We begin by demanding (guided by the form of the solution for $U_0(x, x'; s)$) the solution to be of the form

$$U(x, x'; s) = \exp\left\{-x_\mu A^{\mu\nu}(s) x_\nu - B_\mu(s) x^\mu - C(s)\right\}. \tag{A2}$$

Substituting eqn. (A2) in eqn. (A1) we get the equations for the unknowns A(s), B(s) and C(s) to be

$$
\begin{aligned}
2\frac{\partial \mathrm{A}}{\partial s} &= 4\mathrm{A}^2 + \lambda^2 \\
2\frac{\partial \mathrm{B}}{\partial s} &= 4\mathrm{A}\cdot\mathrm{B} \\
2\frac{\partial \mathrm{C}}{\partial s} &= \mathrm{B}\cdot\mathrm{B} - 2\mathrm{Tr}(\mathrm{A}) + m^2.
\end{aligned} \tag{A3}
$$

The solutions to the above equations are

$$
\begin{aligned}
\mathrm{A}(s) &= -\frac{1}{2}\lambda\cot(2\lambda s + \mathrm{C}_1) \\
\mathrm{B}(s) &= \mathrm{C}_2\mathrm{cosec}(2\lambda s + \mathrm{C}_1) \\
\mathrm{C}(s) &= -\frac{1}{2}\frac{\mathrm{C}_2^2}{\lambda}\cot(2\lambda s + \mathrm{C}_1) - \frac{1}{2}\,\mathrm{Tr}\left[\ln\mathrm{cosec}(2\lambda s + \mathrm{C}_1)\right] + \frac{m^2}{2}s - \frac{1}{2}\ln\mathrm{C}_3
\end{aligned} \tag{A4}
$$

where C_1, C_2 and C_3 are constants with respect to s. Using eqn. (A4) in eqn. (A2) we have

$$
U(x,x';s) = \frac{1}{4\pi^2}e^{-\frac{m^2}{2}s}\left[\frac{(2\pi)^4\mathrm{C}_3}{\det\left(\sin(2\lambda s + \mathrm{C}_1)\right)}\right]^{\frac{1}{2}} e^{l(x,x';s)} \tag{A5}
$$

where

$$
l(x,x';s) = \frac{1}{2}x^2\lambda\cot(2\lambda s + \mathrm{C}_1) - x\mathrm{C}_2\mathrm{cosec}(2\lambda s + \mathrm{C}_1) + \frac{1}{2}\frac{\mathrm{C}_2^2}{\lambda}\cot(2\lambda s + \mathrm{C}_1). \tag{A6}
$$

Using the initial condition $U(x,x';0) = \delta^4(x-x')$ and the identity

$$
\delta(x-x') = \lim_{\sigma\to 0}\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{(x-x')^2}{2\sigma}} \tag{A7}
$$

we can fix C_1, C_2 and C_3 to be

$$
\mathrm{C}_1 = 0 \quad , \qquad \mathrm{C}_2 = \lambda\cdot x' \qquad \text{and} \qquad \mathrm{C}_3 = \frac{1}{(2\pi)^4}\det(\lambda). \tag{A8}
$$

Using eqn. (A8) in eqn. (A5) we have

$$
U(x,x';s) = -i\frac{1}{4\pi^2 s^2}e^{-\frac{m^2}{2}s}\left[\det\left(\frac{\lambda s}{\sin\lambda s}\right)\right]^{\frac{1}{2}} e^{l(x,x';s)} \tag{A9}
$$

where

$$
l(x,x';s) = \frac{1}{2}(x^2 + x'^2)\lambda\cot(\lambda s) - xx'\lambda\mathrm{cosec}(\lambda s). \tag{A10}
$$

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